Pergamon

J. Appl. Maths Mechs, Vol. 61, No. 3, pp. 503–511, 1997

0021–8928/97 $24.00+0.00

PII: S0021–8928(97)00062-2

SOLUTION OF ELASTOPLASTIC PROBLEMS OF THE NON-AXIALLY SYMMETRIC DEFORMATION OF BODIES OF REVOLUTION†

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(*Received 5 June* 1996)

A numerical method of solving spatial problems involving homogeneous isotropic bodies of revolution which obey the equations of flow theory and are under a non-axially symmetric load is developed. The method makes use of an algorithm in which a step is made in the load and iterations are carried out on this step. The Ritz method is used to solve the elastic problem at each iteration. In this method, an expansion in a system of trigonometric functions along a peripheral direction and with respect to the coordinates in the meridian plane is used, that is, a two-dimensional finite-element approximation. It is proved that the iterative process converges in the case of an isotropically hardening body which obeys the associated flow law subject to the Mises plasticity condition. Sufficient conditions for the Ritz solution to converge, on iteration, to the exact solution are also obtained both in the case of external and internal problems. The method is used to calculate a preloaded elastoplastic half-space with a blind hole as the load applied to the lateral surface of the hole is removed. The problem simulates the process of boring a hole in a body which is used to determine the residual stresses in it. A comparison is made between the results obtained and the results of the solution of the same problem in an elastic formulation which is used in practice at the present time to determine residual stresses.

1. THE SYSTEM OF EQUATIONS AND BOUNDARY CONDITIONS

Consider an elastoplastic body of revolution, which occupies a domain Ω in a three-dimensional Euclidean space R^3 and suppose that $\xi^1 = r$, $\xi^2 = \varphi$, $\xi^3 = z$ are the cylindrical coordinates of a point $\mathbf{x}$ of the domain Ω and that t is a loading parameter.

The body is subject to a quasistatic load by the volume forces $\mathbf{X}(\mathbf{x}, t)$ which are distributed in Ω, the surface forces $\mathbf{q}(\mathbf{x}, t)$ which are distributed over the boundary Γ of the domain Ω and the specified displacements $\mathbf{U}(\mathbf{x}, t)$.

We will write the equilibrium and Cauchy equations as

$$\nabla_j\sigma^{ji} + X^i = 0, \quad \varepsilon_{ij} = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i) \tag{1.1}$$

where σ^{ji} and X^i are the components of the stress tensor and of the volume force, and u_i and ε_{ij} are the components of the displacement and of the strain tensor.

We now consider an elastoplastic body which obeys the constitutive equations of flow theory. The stressed state of the body satisfies the condition for plasticity with isotropic hardening

$$f(\sigma^{ij}) - \sigma_Y \leqslant 0 \tag{1.2}$$

where f is a yield function, which is a homogeneous function of the stresses of degree 1 and σ_Y is the yield stress.

The gradient law for the rate of plastic deformation

$$\dot{\varepsilon}_{ij}^p = \dot{\lambda}\varphi_{ij}; \quad \varphi_{ij} = \partial f / \partial\sigma^{ij}, \quad \dot{\lambda} \geqslant 0 \tag{1.3}$$

is satisfied.

The body is subject to strain hardening which can be specified by the following equation

$$\dot{\sigma}_Y = h\sigma^{ij}\dot{\varepsilon}_{ij}^p / \sigma_Y = h\dot{\lambda}\sigma^{ij}\varphi_{ij} / \sigma_Y \tag{1.4}$$

†*Prikl. Mat. Mekh.* Vol. 61, No. 3, pp. 520–528, 1997.

where $h(\sigma_Y)$ is the hardening coefficient ($h = \partial\sigma/\partial\varepsilon_p$) in the case of a uniaxial stressed state).

Finally, the differential form of Hooke's law is

$$\dot{\sigma}^{ij} = D^{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^p_{kl}) = D^{ijkl}(\dot{\varepsilon}_{kl} - \dot{\lambda}\varphi_{ij}) \tag{1.5}$$

where D^{ijkl} are the components of the elasticity constants tensor.

Equations (1.2)–(1.5) form the system of constitutive equations of flow theory. The stresses and strains are functions of the point on the body **x** and the loading parameter t.

We differentiate relation (1.2), with the equality sign, with respect to t, and take account of (1.3)–(1.5). We then obtain

$$\dot{\lambda} = \begin{cases} \dfrac{\vartheta^{ij}\dot{\varepsilon}_{ij}}{\varphi_{ij}\vartheta^{ij} + h}, & f(\sigma^{ij}) = \sigma_Y \cap \vartheta^{ij}\dot{\varepsilon}_{ij} \geqslant 0 \\ 0, & f(\sigma^{ij}) < \sigma_Y \cup \vartheta^{ij}\dot{\varepsilon}_{ij} < 0 \end{cases} \tag{1.6}$$

where $\vartheta^{ij} = D^{ijkl}\varphi_{kl}$.

Equations (1.4) and (1.5), in the case when $\varepsilon_{ij}(\mathbf{x}, t)$ is specified at each point **x**, form a system of ordinary differential equations in σ_Y and σ^{ij}. The final form of Eq. (1.5) will be

$$\dot{\sigma}^{ij} = A^{ijkl}\dot{\varepsilon}_{kl}; \quad A^{ijkl} = D^{ijkl} - H^{ijkl} \tag{1.7}$$

$$H^{ijkl} = \begin{cases} \dfrac{\vartheta^{ij}\vartheta^{kl}}{\varphi_{mn}\vartheta^{mn} + h}, & f = \sigma_Y \cap \vartheta^{ij}\dot{\varepsilon}_{ij} \geqslant 0 \\ 0, & f < \sigma_Y \cup \vartheta^{ij}\dot{\varepsilon}_{ij} < 0 \end{cases}$$

Equations (1.1), (1.4) and (1.7) form the complete system of equations for the problem under consideration.

It remains to state the initial and boundary conditions of the problem.

When $t = 0$, we have $\sigma^{ij} = \sigma_0^{ij}$, $\sigma_Y = \sigma_Y^0$, where σ_0^{ij}, σ_Y^0 are specified functions of **x**. It may be assumed, without loss of generality, that $u_i = 0$ when $t = 0$.

In the part Γ_q of the boundary Γ of the domain Ω where the surface forces $\mathbf{q}(\mathbf{x}, t)$ are specified, the boundary conditions are written in the form

$$\sigma^{ji}\nu_j = q^i \tag{1.8}$$

where ν_i are the components of the normal to the boundary surface and q^i are the components of the surface force.

The displacements $\mathbf{U}(\mathbf{x}, t)$ are specified on the other part of the boundary Γ_u and the boundary conditions have the form

$$u_i = U_i \tag{1.9}$$

where U_i are the components of the specified displacements.

On the axis of rotation ($r = 0$), it is necessary to specify conditions which express the situation that one and the same point corresponds to different values of the angle φ

$$u_r = A(z)\cos\varphi + B(z)\sin\varphi \tag{1.10}$$
$$u_\varphi = -A(z)\sin\varphi + B(z)\cos\varphi, \quad u_z = C(z)$$

where u_r, u_φ, u_z are the physical components of the displacement vector in a cylindrical system of coordinates, that is, $u_r = u_1$, $u_\varphi = u_2/r$, $u_z = u_3$.

The body can occupy an unbounded domain. We shall assume that the volume and surface forces are applied in a bounded part of the body. Then, in order for the solution of the problem to be unique, the following conditions can be imposed at infinity [1]

$$(u_r^2 + u_\varphi^2 + u_z^2)^{1/2} = O(R^{-1}), \quad \partial\chi_i / \partial x_j = o(R^{-1}); \quad R = (r^2 + z^2)^{1/2} \tag{1.11}$$

where, in the second formula, the components of the displacements χ_i and the derivatives are considered in the Cartesian system of coordinates $x_1 = x, x_2 = y, x_3 = z$.

Finally, it is necessary to impose the periodicity conditions

$$u_i(r,\varphi+2\pi,z) = u_i(r,\varphi,z) \tag{1.12}$$

2. THE STEPWISE METHOD FOR SOLVING THE SYSTEM OF EQUATIONS

We subdivide the range of variation of the loading parameter t into N parts and denote t at the nodes by t_p ($p = 0, 1, \ldots, N$). The volume and surface forces and the specified displacements when $t = t_p$ are denoted by X_p^i, q_p^i, U_i^p, respectively. At each step $p = 1, \ldots, N$ of the change in the above-mentioned quantities, we set up the following system of equations

$$\nabla_j \sigma^{ji} + X_p^i = 0, \quad \sigma^{ij} = \sigma_{p-1}^{ij} + A_{p-1}^{ijkl} \Delta\varepsilon_{kl}^p \tag{2.1}$$

$$\Delta\varepsilon_{ij}^p = \frac{1}{2}(\nabla_i(\Delta u_j^p) + \nabla_j(\Delta u_i^p))$$

Here

$$A_p^{ijkl} = D^{ijkl} - H_p^{ijkl}$$

$$H_p^{ijkl} = \begin{cases} \dfrac{D^{ijst}\varphi_{st}^p \varphi_{mn}^p D^{mnkl}}{\varphi_{mn}^p D^{mnst} \varphi_{st}^p + h^p}, & f = \sigma_Y \\ 0, & f < \sigma_Y \end{cases}$$

and the superscript p on a quantity denotes that it has been calculated at the corresponding value of the loading parameter.

To system (2.1), we add the boundary conditions

$$\sigma^{ij}\nu_j|_{\Gamma_q} = q_p^i; \quad \Delta u_i^p|_{\Gamma_u} = \Delta U_i^p \tag{2.2}$$

We treat the system of equations (2.1) with boundary conditions (2.2), the conditions on the axis, the conditions at infinity and the periodicity conditions as a system in Δu_i^p. Since A^{ijkl} is independent of ε_{ij}, which has been assumed in (2.1), this system can be considered as a system of equations for a certain inhomogeneous anisotropic elastic body with additional volume and surface forces.

After the system of equations has been solved, the displacement is represented in the form

$$u_i = u_i^{p-1} + \frac{t - t_{p-1}}{t_p - t_{p-1}} \Delta u_i^p \tag{2.3}$$

The strains ε_{ij} are determined using the displacements from the Cauchy equations, and these strains are then substituted into the constitutive equations (1.4) and (1.7), from where $\sigma_p^{ij} = \sigma^{ij}(t_p)$ and $\sigma_Y^p = \sigma_Y(t_p)$ are obtained when the initial conditions $\sigma^{ij}(t_{p-1}) = \sigma_{p-1}^{ij}$ and $\sigma_Y(t_{p-1}) = \sigma_Y^{p-1}$ are satisfied.

3. AN ITERATIVE METHOD OF SOLVING THE SYSTEM OF EQUATIONS ON A STEP IN THE LOADING

We will solve the system of equations (2.1) with boundary conditions (2.2) using an iterative method which is analogous to the method of supplementary stresses in the deformation theory of plasticity [2]. The second equation of (2.1) is written in the form

$$\sigma_{(n)}^{ij} = \sigma_{p-1}^{ij} + D^{ijkl}\Delta\varepsilon_{kl}^{(n)} - \Delta\alpha_{(n-1)}^{ij} \tag{3.1}$$

$$\Delta\alpha_{(n)}^{ij} = H_{p-1}^{ijkl}\Delta\varepsilon_{kl}^{(n)}, \quad \Delta\alpha_{(0)}^{ij} = 0$$

where $n = 1, 2, \ldots$ is the number of iterations on the pth step of the change in the loading parameter. Then, on the nth iteration, the system of equations and the boundary conditions from Section 2 are written in the form

$$\begin{aligned}
&\nabla_j(D^{jikl}\Delta\varepsilon_{kl}^{(n)}) + \nabla_j\sigma_{p-1}^{ji} - \nabla_j(\Delta\alpha_{(n-1)}^{ji}) + X_p^i = 0 \\
&\Delta\varepsilon_{ij}^{(n)} = \frac{1}{2}(\nabla_i(\Delta u_j^{(n)}) + \nabla_j(\Delta u_i^{(n)})) \\
&D^{jikl}\Delta\varepsilon_{kl}^{(n)}\nu_j|_{\Gamma_q} = q_p^i - \sigma_{p-1}^{ji}\nu_j|_{\Gamma_q} + \Delta\alpha_{(n-1)}^{ji}\nu_j|_{\Gamma_q} \\
&\Delta u_i^{(n)}|_{\Gamma_u} = \Delta U_i^p
\end{aligned} \tag{3.2}$$

and represent the system of equations and the boundary conditions for an elastic body with the same elasticity constants D^{ijkl} as the initial body but with changed volume and surface forces.

4. ON THE CONVERGENCE OF THE ITERATIVE METHOD

The proposed iterative method belongs to the Il'yushin class of methods of elastic solutions, the proof of the convergence of which is given in [3] in the case of the deformation theory of plasticity. Sufficient conditions for the convergence of the iterative method for the general case of a non-elastic material have been given in [4]. Starting from these conditions, for iterations (3.2) to converge to the solution of system (2.1)–(2.2) it is sufficient that

$$\gamma_1 D^{ijkl}\varepsilon_{ij}\varepsilon_{kl} \leqslant A^{ijkl}\varepsilon_{ij}\varepsilon_{kl} \leqslant \gamma_2 D^{ijkl}\varepsilon_{ij}\varepsilon_{kl} \tag{4.1}$$

for any symmetric tensor $\boldsymbol{\varepsilon}$ where

$$\gamma_1 > 0, \quad \gamma_2 < 2 \tag{4.2}$$

In the case of an isotropic body and a Mises flow function $f(\sigma^{ij}) = \surd(3/2)(s^{ij}s_{ij})^{1/2}$ (s_{ij} are the components of the stress deviator) we shall determine for which values of the elastoplastic constants conditions (4.1) and (4.2) are satisfied.

From (1.7), we have

$$A^{ijkl} = \lambda g^{ij}g^{kl} + \mu(g^{ik}g^{il} + g^{il}g^{ik}) - \frac{9\mu^2}{h+3\mu}\frac{s^{ij}s^{kl}}{f^2}$$

where λ and μ are Lamé parameters and g^{ij} are the components of the metric tensor.

We expand the expressions in (4.1)

$$D^{ijkl}\varepsilon_{ij}\varepsilon_{kl} = \lambda(\varepsilon_i^i)^2 + 2\mu\varepsilon^{ij}\varepsilon_{ij}$$

$$A^{ijkl}\varepsilon_{ij}\varepsilon_{kl} = \lambda(\varepsilon_i^i)^2 + 2\mu\varepsilon^{ij}\varepsilon_{ij} - \frac{9\mu^2}{h+3\mu}\frac{(s^{ij}\varepsilon_{ij})^2}{f^2}$$

Analysis of the above expressions gives

$$\gamma_1 = h/(h+3\mu), \quad \gamma_2 = 1$$

and conditions (4.2) are satisfied when $h > 0$, that is, the material is hardening.

5. USE OF THE RITZ METHOD TO SOLVE THE SYSTEM OF EQUATIONS ON AN ITERATION

In order to write down the Lagrange variational principle which is equivalent to system (3.2), we subtract the equilibrium equation and the static boundary conditions when $t = 0$ from the first and third equations of (3.2) respectively. This procedure is necessary in order that integrals along a boundary at infinity should not occur in the case of an infinite domain and non-zero stresses at infinity in the Lagrangian functional. The Lagrangian functional, corresponding to (3.2), is obtained by means of some straightforward operations

$$J=\frac{1}{2}\int_{\Omega} D^{ijkl}\Delta\varepsilon_{ij}^{(n)}\Delta\varepsilon_{kl}^{(n)}d\Omega+\int_{\Omega}(\sigma_{p-1}^{ij}-\sigma_0^{ij}-\Delta\alpha_{(n-1)}^{ij})\Delta\varepsilon_{ij}^{(n)}d\Omega-$$
$$-\int_{\Omega}(X_p^i-X_0^i)\Delta u_i^{(n)}d\Omega-\int_{\Gamma_q}(q_p^i-q_0^i)\Delta u_i^{(n)}d\Gamma \tag{5.1}$$

where $\sigma_0^{ij}, X_0^i, q_0^i$ are the components of the stresses and the volume and surface forces when $t = 0$.

The Ritz method is used to minimize (5.1).

We now introduce the vector notation which is conventionally used in the finite-element method

$$\mathbf{d}^T=[u_r,u_\varphi,u_z],\quad \mathbf{q}^T=[q_r,q_\varphi,q_z],\quad \mathbf{X}^T=[X_r,X_\varphi,X_z]$$
$$\boldsymbol{\epsilon}^T=[\varepsilon_r,\varepsilon_\varphi,\varepsilon_z,\gamma_{r\varphi},\gamma_{rz},\gamma_{\varphi z}],\quad \boldsymbol{\sigma}^T=[\sigma_r,\sigma_\varphi,\sigma_z,\tau_{r\varphi},\tau_{rz},\tau_{\varphi z}]$$
$$\Delta\boldsymbol{\alpha}^T=[\Delta\alpha_{rr},\Delta\alpha_{\varphi\varphi},\Delta\alpha_{zz},\Delta\alpha_{r\varphi},\Delta\alpha_{rz},\Delta\alpha_{\varphi z}]$$

where the physical components of the tensors in a cylindrical system of coordinates are enclosed in square brackets.

We will approximate the displacement vector in the form

$$\mathbf{d}(r,\varphi,z)=\sum_{k=0}^{n}(\mathbf{a}^k(r,z)\cos k\varphi+\mathbf{b}^k(r,z)\sin k\varphi) \tag{5.2}$$

where $\mathbf{a}^k$ and $\mathbf{b}^k$ are vector functions with components with respect to r, φ and z.

It should be noted that, in order for approximation (5.2) to be applicable, it is necessary that the surfaces Γ_u and Γ_q should represent surfaces of revolution of parts of the boundary of the meridian cross-section about the z axis. In order to approximate the coefficients of $\mathbf{a}^k$ and $\mathbf{b}^k$, we subdivide a certain bounded part of the meridian cross-section into finite elements while the remaining part is subdivided into infinite elements [5]. Then

$$\mathbf{a}^k=\sum_{i=1}^{m}G_i\mathbf{a}_i^k;\quad \mathbf{b}^k=\sum_{i=1}^{m}G_i\mathbf{b}_i^k \tag{5.3}$$

where G_i is a function of the form of the ith mesh point of the meridian cross-section in the finite and infinite elements, $\mathbf{a}_i^k$ and $\mathbf{b}_i^k$ are the values of the expansion coefficients at the ith mesh point and m is the number of mesh points in the meridian cross-section.

When a radial infinite element is used [5], which is formed by the side of an adjoining finite element and two rays which pass through the origin of coordinates, the shape functions are represented as

$$G_i=\overline{G}_iR_0^l\,/\,R^l$$

where $\overline{G}_i$ is the shape function of the ith mesh point in a finite element which is adjacent to an infinite element, calculated at the point of intersection of the common side of the elements with a ray which joins the origin of coordinates and the point of the infinite element under consideration: R and R_0 are the distances from the origin of coordinates to the point of the infinite element and the point at which the functions $\overline{G}_i$ is calculated respectively and l is the order of the asymptotic form of the displacement at infinity with respect to R.

Expressions (5.2) and (5.3) can be written in the vector–matrix form which is accepted in the finite-element method

$$\mathbf{d}=\mathbf{G}\boldsymbol{\delta} \tag{5.4}$$

where $\boldsymbol{\delta}$ is the vector formed by the components $\mathbf{a}_i^k$, $\mathbf{b}_i^k$ $(k = 0,\ldots,n; i = 1,\ldots,m)$ and $\mathbf{G}$ is the matrix of the form functions.

The strains are also expressed in terms of the vector $\boldsymbol{\delta}$

$$\boldsymbol{\epsilon}=\mathbf{B}\boldsymbol{\delta} \tag{5.5}$$

where $\mathbf{B}$ is the matrix of the gradients.

The form function and gradient matrices consist of the submatrices $\mathbf{G}_k$ and $\mathbf{B}_k$ $(k = 0, \ldots, n)$ corresponding to different harmonics. The elements of the matrices $\mathbf{G}_k$ and $\mathbf{B}_k$ are represented by the products of certain functions of r and z with $\cos k\varphi$ or $\sin k\varphi$.

Using the notation which has been adopted here, Hooke's law can be written in the form

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon} \tag{5.6}$$

where $\mathbf{D}$ is a symmetric 6×6 matrix, the coefficients of which are the physical components of the elasticity constants tensor in a cylindrical system of coordinates r, φ and z.

The Ritz method which has been described reduces the problem of minimizing functional (5.1) to solving the following system of linear algebraic equations in the increments of the mesh point coefficients $\boldsymbol{\delta}$

$$\begin{aligned} &\left\{\int_\Omega \mathbf{B}^T \mathbf{D}\mathbf{B} d\Omega\right\}\Delta\boldsymbol{\delta}^{(n)} + \int_\Omega \mathbf{B}^T(\boldsymbol{\sigma}_{p-1} - \boldsymbol{\sigma}_0 - \Delta\boldsymbol{\alpha}_{(n-1)})d\Omega - \\ &-\int_\Omega \mathbf{G}^T(\mathbf{X}_p - \mathbf{X}_0)d\Omega - \int_\Gamma \mathbf{G}^T(\mathbf{q}_p - \mathbf{q}_0)d\Gamma = 0 \end{aligned} \tag{5.7}$$

By virtue of the orthogonality of the system of trigonometric functions in the interval $0 \leqslant \varphi \leqslant 2\pi$, system (5.7) decomposes into separate systems of equations for each harmonic. However, unlike the case of an elastic body, on the right-hand sides of these systems there are not only coefficients corresponding to the expansion of the external load in a system of trigonometric functions but, also, coefficients corresponding to the expansion of the stresses. Hence, even if the load has a finite number of harmonics, the required displacement will have an infinite number of harmonics in the general case.

We note that, in solving system (5.7), it is necessary to take account of the boundary conditions in the displacements and the conditions on the z axis which reduce to simple conditions for the increments in the components of the vectors $\mathbf{a}^k$ and $\mathbf{b}^k$.

We will now consider the case when one and the same system of functions of the Ritz method is used in each iteration. Then, (5.7) is rewritten in the form

$$\boldsymbol{\psi}^{(n-1)} + \left\{\int_\Omega \mathbf{B}^T \mathbf{D}\mathbf{B} d\Omega\right\}(\Delta\boldsymbol{\delta}^{(n)} - \Delta\boldsymbol{\delta}^{(n-1)}) = 0 \tag{5.8}$$

where $\boldsymbol{\psi}^{(n)}$ is the residual of system (5.7) after the nth iteration, which is defined by the expression

$$\begin{aligned} &\boldsymbol{\psi}^{(0)} = \int_\Omega \mathbf{B}^T(\boldsymbol{\sigma}_{p-1} - \boldsymbol{\sigma}_0)d\Omega - \int_\Omega \mathbf{G}^T(\mathbf{X}_p - \mathbf{X}_0)d\Omega - \int_\Omega \mathbf{G}^T(\mathbf{q}_p - \mathbf{q}_0)d\Gamma \\ &\boldsymbol{\psi}^{(n)} = \int_\Omega \mathbf{B}^T(\Delta\boldsymbol{\alpha}_{(n-1)} - \Delta\boldsymbol{\alpha}_{(n)})d\Omega, \quad n = 1, 2, \ldots \end{aligned} \tag{5.9}$$

Representation (5.8), (5.9) is convenient due to the fact that, in order to calculate the residual, it is only necessary to carry out an integration over that part of Ω which is in a plastic state.

Additional advantages associated with the use of one and the same system of functions of the Ritz method include the constancy of the stiffness matrix of the system of equations at each iteration. This enables us to re-order and decompose the matrix prior to the iterative process and solve only triangular systems at each iteration [6].

6. ON THE CONVERGENCE OF THE RITZ SOLUTION TO THE EXACT SOLUTION IN AN ITERATION

Let Δ be the diameter of the two-dimensional domain, which is subdivided into discrete areas by the finite elements, and let β be the maximum diameter of the finite elements. As previously, the parameter n denotes the number of harmonics in the expansion of displacements (5.2). Suppose $\mathbf{d}$ is the exact solution of the system of equations in an iteration and that $\mathbf{d}^r$ is the Ritz solution.

Theorem. Suppose that the following conditions are satisfied

1. $\mathbf{d}$ is a doubly continuously differentiable function with respect to r, φ and z;

2. when $R \Rightarrow \infty$, we have $d_i = O(R^{-1}, \partial d_i/\partial x_j = O(R^{-2}), \partial^2 d_i/\partial x_j \partial x_k = O(R^{-3})$, where d_1, d_2 and d_3 are the components of $\mathbf{d}$ in a Cartesian system of coordinates x_1, x_2, x_3, R is the distance to the coordinate origin and $i, j, k = 1, 2, 3$;

3. in the meridian cross-section, each finite element with a diameter γ contains a circle of diameter $\tau\gamma$, where τ is a positive number specified in advance;

4. the boundary of the domain in the meridian cross-section which is discretized by the finite elements is piecewise once smooth;

5. $R_{\min}/\Delta > B$, where $R_{\min}$ is the distance from the origin of coordinates to the boundary of the finite element domain (see condition 4 of the theorem) and B is a positive number specified in advance.

The inequality

$$\|\mathbf{d}-\mathbf{d}^r\| \leqslant \|\mathbf{d}-\mathbf{d}_1\| \tag{6.1}$$

then holds, where $\| \cdot \|$ is the energy norm and $\mathbf{d}_1$ is a certain vector function which satisfied the equality

$$\lim_{n \Rightarrow \infty} \lim_{\Delta \Rightarrow \infty} \lim_{\beta \Rightarrow 0} \|\mathbf{d}-\mathbf{d}_1\| = 0 \tag{6.2}$$

7. SOLUTION OF PROBLEM ON THE DEFORMATION OF AN ELASTOPLASTIC HALF-SPACE WITH A HOLE OR A BORE

We now consider the problem of the deformation of an elastoplastic isotropic half-space with a blind hole when the surface load applied to the lateral surface of the hole is removed. This problem simulates the process in which a hole is drilled for determining the residual stresses at a point on the body surface [7].

We construct a Cartesian system of coordinates with the origin at the specified point, with its x and y axes lying in the boundary plane of the half-space and the z axis perpendicular to the first two axes (Fig. 1). At the initial instant of time, we specify a homogeneous stressed state in the body with components in the cylindrical system of coordinates r, φ and z

$$\begin{gathered}\sigma_r = \sigma_1 \cos^2\varphi + \sigma_2 \sin^2\varphi, \quad \sigma_\varphi = \sigma_1 \sin^2\varphi + \sigma_2 \cos^2\varphi \\ \tau_{r\varphi} = \frac{\sigma_2-\sigma_1}{2}\sin 2\varphi, \quad \sigma_z = \tau_{rz} = \tau_{\varphi z} = 0\end{gathered} \tag{7.1}$$

where σ_1 and σ_2 are the principal components of the residual stresses, the principal axes of which are directed along the x and y axes.

We will now consider the boundary conditions of the problem. It is assumed that the surface load on the lateral surface falls from the initial value corresponding to the stressed state (7.1) to zero such that, at a certain intermediate instant of time, it has the following components

$$\begin{aligned}&q_r = q_\varphi = q_z = 0; \quad -z_2 \leqslant z < 0 \\ &q_r = -\sigma_1 \cos^2\varphi - \sigma_2 \sin^2\varphi, \\ &q_\varphi = \sigma_1 \sin\varphi\cos\varphi - \sigma_2 \sin\varphi\cos\varphi \\ &q_z = 0; \quad -z_0 \leqslant z < -z_2\end{aligned} \tag{7.2}$$

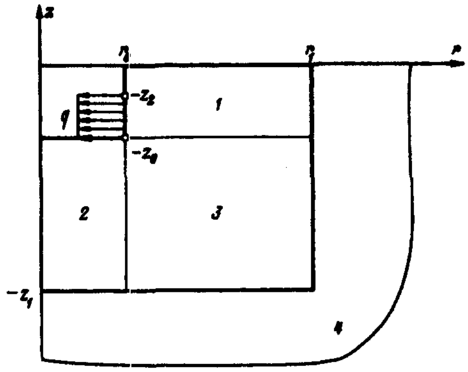

Fig. 1.

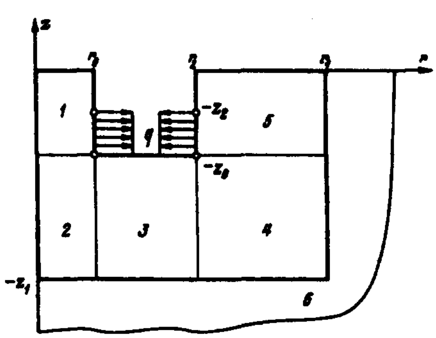

Fig. 2.

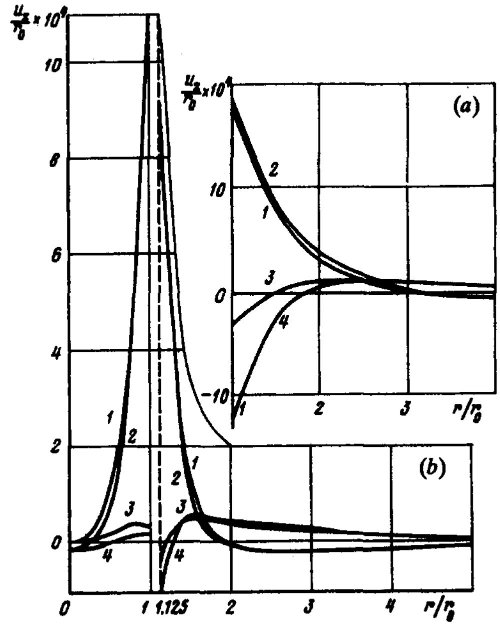

Fig. 3.

where at the initial instant $z_2 = 0$, and at the final instant $z_2 = z_0$.

The bottom of the cavity and the remaining boundaries of the body are stress-free. When the hole is bored, the body is deformed elastically at infinity. Hence, the displacement at infinity has the same asymptotic form as in the elastic problem which, as has been shown in [8], is R^{-2}, where R is the distance from the origin of coordinates.

On the axis of revolution, u_z is independent of φ and, as a result of the symmetry of the problem with respect to the coordinate planes xOz and yOz, $u_r = u_\varphi = 0$.

In the case of an annular bore (Fig. 2), the load (7.2) is applied to the external part of the lateral surface of the bore while a load which is the opposite of (7.2) is applied to its internal part.

For the calculations, we consider a material with an elastic modulus $E = 7 \times 10^4$ MPa, a yield point $\sigma_Y = 280$ MPa, a modulus of strain hardening $h = 70$ MPa and a Poisson's ratio $\nu = 0.3$. The initial stresses have the principal components $\sigma_1 = 240$ MPa and $\sigma_2 = 0$.

The problems are solved by the stepwise-iterative method described in Sections 2, 3 and 5. The required displacements are approximated by formula (5.2) in which only even harmonics occur by virtue of the symmetry of the problem with respect to the xOz and yOz coordinate planes.

In the case of a half-space with a hole, the depth of this hole is equal to its radius. The part of the meridian cross-section which is discretized by the finite elements has a radius $r_1 = 5r_0$ and a height $z_1 = 5r_0$, where r_0 is the radius of the hole. This domain is divided up into three zones (Fig. 1) and each zone is divided up into finite elements. The remaining part of the meridian cross-section (zone 4) is divided up into infinite elements. The whole of the meridian cross-section contains 300 finite elements and 40 infinite elements. The finite element mesh is compressed in the direction towards the hole. The load applied to the lateral surface of the hole is subdivided into five different steps with respect to the parameter z_2. The same harmonics are retained as in the case of the hole.

To check the accuracy of the solutions obtained, calculations were carried out using different values of the parameters in the numerical model. The results of these additional calculations differed from the main calculations by a few percent.

Graphs of the displacements u_z, divided by the radius of the hole, as a function of r on the boundary of a half-space with a hole (a) and of the displacements u_z, divided by the internal radius of the bore, on the boundary of a half-space inside and outside of annular bore (b) are shown in Fig. 3. The elastic and elastoplastic solutions when $\varphi = 0$ are denoted by the numbers 1 and 2 and the same solutions when $\varphi = \pi/2$ are denoted by 3 and 4. A characteristic feature of the elastoplastic solution is that the displacement u_z on the edge of the hole when $\varphi = \pi/2$ is comparable with the displacement u_z when $\varphi = 0$. It can be seen that the elastic and elastoplastic solutions are much closer to each other in the case of an annular bore than in the case of a hole.

The results show that the determination of residual stresses from the displacements of points of a body in the neighbourhood of an aperture can lead to large errors when it is assumed that the problem is an elastic one. The use of an annular bore in this case leads to much smaller errors.

This research was carried out with financial support from the Russian Foundation for Basic Research (1996-01-01538).

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Translated by E.L.S.